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Natural frequency intervals for vibrating systems with polytopic uncertainty ☆

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ABSTRACT

We consider the problem of locating the natural frequencies of uncertain systems whose describing matrices are functions of an unknown parameter vector which is included in an assigned bounding set. We face what we call the *weak frequency interval detection problem* of determining the smallest interval which includes all possible frequencies. We show that if the system matrices depend affinely on the parameter vector, whose bounding set is a compact polyhedron, then this problem requires the solution of a finite number of eigenvalue problems associated with the vertices of such a polyhedron. Unfortunately, detecting the intervals associated with all the natural frequencies (*strong frequency interval detection problem*), cannot rely on this property, so that one must resort to Monte Carlo methods or numerical optimization to find them. We show that the strong version is solvable "exploring the vertices only" under some stronger assumptions. In the case in which the uncertainty bounding set is not defined by linear inequalities, not even the extremal frequencies can be associated with the vertices of the admissibility domain. Then again, numerical approach is necessary unless we accept to merge the original system in a larger one of an "affine nature". Finally, we present as an application the study of structures with uncertain mass distribution.

1. Introduction

Parameter indeterminacy is a phenomenon which undermines system analysis in engineering. The uncertainty which often occurs in the parameters is indeed a serious problem, especially when sophisticated mathematical tools and software have to be applied. Uncertainty in the parameters may be caused by unpredictable physical variations, inaccurate modeling due to approximation or it can be just the consequence of the lack of data in the analysis or design stage. In vibrating mechanical structures, uncertainty often originates from the fact that the load and then the mass is not always known a priori or it may change, or by the fact that some structures are re-configurable, and then stiffness coefficients are not fixed.

The "perturbation method" is a typical approach in this framework. It is based on the assumption that the parameters can vary in a small neighborhood of their nominal value [1,2]. The assumption of small perturbation is not reasonable in several cases, so that the unknown-but-bounded approach can be more effective. Essentially, the basic assumption is that the parameters are unknown but bounded in a region which can be reasonably inferred by practical experience, a priori

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information or technical constraints. In the case of vibrating systems, the fundamental problem is that of determining the intervals in which the natural frequencies may range once the admissible parameter set is given.

In this framework we acknowledge the contribution of Qiu and co-authors. In [3] the problem of determining frequency bounds for interval vibrating systems has been introduced. A method based on the Rayleigh iteration has been proposed. A refinement of the bounds has been proposed in [4] and more recently in [5,6]. The main result shows that under appropriate assumptions on the perturbation matrices it is possible to compute exactly the admissible frequency intervals. In dealing with systems with interval uncertainties, the interval arithmetic offers effective tools which have been successfully exploited in [4,7,8]. A so-called interval-sensitivity method has been proposed in [9] and bounds for the frequency intervals are determined via optimization algorithms. An approach based on fuzzy logic to evaluate the frequency response has been proposed in [10]. Finally, in [11] it has been shown that the exact frequency intervals can be determined if the perturbation data satisfy certain assumption.

With very few exceptions, the uncertainty considered in the existing work is of the interval type. This basically means that each parameter of the system matrices is upper and lower bounded, independently of the others. Although this assumption is appropriate in most cases, there are many systems which do not fit in this class. Clearly it is always possible to embed the uncertain system in a larger one of an interval nature. The price we pay is that we get an over-estimation of the frequency intervals. For instance the family of matrices

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix},$$

with $1 \le k_i \le 2$ is a subset of the family

$$\hat{\mathbf{K}} = \begin{bmatrix} a & -b \\ -b & c \end{bmatrix},$$

with $2 \le a \le 4$, $1 \le b \le 2$, $1 \le c \le 2$. Note that these bounds on a, b, c are the smallest we must fix to include every possible $\mathbf{K}$ in the family $\hat{\mathbf{K}}$. Still the eigenvalues of $\mathbf{K}$ form intervals which are proper subsets of the frequency intervals of the family $\hat{\mathbf{K}}$. Easy computations show that this inclusion introduces a strong over-estimation of the intervals. For instance the smallest value for the first eigenvalue of $\mathbf{K}$ is 0.382, while the smallest value for the first eigenvalue of $\hat{\mathbf{K}}$ is negative -0.56155. Then $\mathbf{K}$ is positive-definite for each value of the parameters while $\hat{\mathbf{K}}$ is not.

In this paper we generalize the existing work to the case in which the uncertainty in the matrices is of the polytopic nature. This means that the compact region including the matrix parameters is defined by linear inequalities, namely it is a polytope. As it is known, a polytope admits an equivalent vertex representation which is more convenient to develop our theory.

The main result of the paper basically shows that for any polytopic system it is possible to find exactly the largest and the smallest natural frequencies of the family or, in other words, the smallest interval including all possible natural frequencies. This problem will be referred to as *weak frequency interval detection*. These extremal frequencies are found by computing the natural frequencies of a finite numbers of "extremal systems" corresponding to the vertices of the polytopic representation and computing the minima and, respectively, the maximal ones over such a set. According to the existing literature [12–14], this is called a "vertex type of result". We stress that this kind of properties are not always quite obvious. For instance a system of first-order linear differential equations with uncertainties of the type considered here does not have the extremal property as far as asymptotic stability (i.e. the system describing matrix having negative real part eigenvalues) is concerned [13]. However, it has been shown [12] that under the a priori assumption that the eigenvalues are real, vertex results hold, for interval types of matrices. It is then reasonable to expect "vertex result" in vibrating systems, since they are characterized by second-order equations whose describing matrices have real eigenvalues.

Unfortunately, even for the polytopic uncertainty description, there are no extreme point results for the strong interval detection problem. In other words, the determination of the smallest interval for all natural frequencies is not solvable by means of the computation of the frequency of a finite number of structures. Then Monte Carlo or numerical optimization seem the only resort. Note that in the case of systems with "large" uncertainties, the fact that only the weak version of the problem can be solved efficiently does not seem to be a crucial issue since the frequency intervals are, in general, deeply overlapping, so that it is often useless to detect exactly all of them while the detection of the global interval of natural frequencies can play a fundamental role.

The paper is structured as follows. In Section 2 we introduce and motivate the class of models. In Section 3 we introduce the main result. In Section 4 we briefly consider more general types of uncertainty for which the weak problem can be solved and we consider the problem of embedding nonlinear uncertainty region in linearly constrained uncertain region to apply our result. In Section 5 we consider a complementary problem. Instead of detecting the smallest interval including all frequencies for all possible values of the parameters, we find the values of the parameters which minimizes the maximal frequency, respectively, maximizes the minimal frequency. This problem is of interest to design a structure to limit either the upper of the lower frequency range. It can be solved via convex optimization for which efficient algorithms are available. We finally propose an application of the theory to the weak interval detection of structures with uncertain mass distribution. We show that the "extremal distributions" which give the extremal frequency are achieved by concentrating all the mass on a single degree of freedom.

2. Model description

In this paper we consider uncertain vibrating systems of the form

$$\mathbf{M}(\mathbf{p})\ddot{\mathbf{q}}(t) = -\mathbf{K}(\mathbf{p})\mathbf{q}(t), \tag{1}$$

where

- $\mathbf{q}(t) \in \mathbb{R}^m$ is the Lagrangian coordinate vector;
- $\mathbf{M}(\mathbf{p})$ is the mass matrix;
- $\mathbf{K}(\mathbf{p})$ is the stiffness matrix;
- $\mathbf{p} \in \mathcal{P}$ is the uncertain parameter vector;
- $\mathcal{P} \subset \mathbb{R}^s$ is the assigned parameter region.

The following assumptions will be considered in the paper:

(A1) $\mathbf{M}(\mathbf{p})$ and $\mathbf{K}(\mathbf{p})$ are symmetric and they are positive definite and positive semi-definite, respectively, for each $\mathbf{p} \in \mathcal{P}$;
(A2) $\mathbf{p}$ is constant in time and there is no other information beside the fact it belongs to set $\mathcal{P}$.

It is well known that, for a fixed $\mathbf{p}$, the natural frequencies of the system are given by the square roots of the eigenvalues of the pair $(\mathbf{M}(\mathbf{p}), \mathbf{K}(\mathbf{p}))$ and, precisely, by the values ω such that

$$[\omega^2\mathbf{M}(\mathbf{p}) - \mathbf{K}(\mathbf{p})]\mathbf{v}(\mathbf{p}) = 0$$

admits a non-trivial solution $\mathbf{v}(\mathbf{p}) \neq 0$. In the sequel, for notational purposes, we use the compact expression $[\mathbf{M}, \mathbf{K}]$ to denote the rectangular $m \times 2m$ matrix achieved by combining $\mathbf{M}$ and $\mathbf{K}$ and we denote by

$$\Sigma[\mathbf{M}(\mathbf{p}), \mathbf{K}(\mathbf{p})] = \{\omega_1(\mathbf{p}), \omega_2(\mathbf{p}), \ldots, \omega_m(\mathbf{p})\}$$

the set of the eigenvalues of $\omega^2\mathbf{M}(\mathbf{p}) - \mathbf{K}(\mathbf{p})$. These eigenvalues will be always considered in their increasing order

$$0 \leq \omega_1(\mathbf{p}) \leq \omega_2(\mathbf{p}) \leq \cdots \leq \omega_m(\mathbf{p}). \tag{2}$$

The admissible intervals $\mathcal{I}_i$ for the ith eigenvalue are defined as follows:

$$\mathcal{I}_i = [a_i, b_i],$$

where

$$a_i = \min_{p \in \mathcal{P}} \quad \omega_i(\mathbf{p}), \tag{3}$$

$$b_i = \max_{p \in \mathcal{P}} \quad \omega_i(\mathbf{p}). \tag{4}$$

Note that the order (2) is valid for fixed $\mathbf{p}$, and therefore the intervals $\mathcal{I}_i$ may have non-empty intersections. In practice we know that $a_i(\mathbf{p}) \leq a_{i+1}(\mathbf{p})$ and that $b_i(\mathbf{p}) \leq b_{i+1}(\mathbf{p})$ but it may well happen that $a_{i+1}(\mathbf{p}) < b_i(\mathbf{p})$. The first problem we consider is the strong version of the root interval detection.

Problem 1. Strong root interval detection. Given $\mathbf{M}(\mathbf{p})$, $\mathbf{K}(\mathbf{p})$ and $\mathcal{P}$, find the intervals $\mathcal{I}_i$.

Unfortunately, in general this problem is not easily solvable even under linearity assumption on $\mathbf{M}(\mathbf{p})$ and $\mathbf{K}(\mathbf{p})$, unless we resort to brute-force computation or randomized algorithms (see for instance [15]). The essential reason is that the "internal" eigenvalues lie in intervals whose extrema can be everywhere in the admissible set. This is not the case for the minimum and maximum eigenvalues. Therefore we consider, accordingly, the weaker version of the problem.

Problem 2. Weak root interval detection. Given $\mathbf{M}(\mathbf{p})$, $\mathbf{K}(\mathbf{p})$ and $\mathcal{P}$, find the smallest interval containing all $\mathcal{I}_i$, namely $[a, b] = [a_1, b_m]$.

2.1. Polytopic uncertainty description

The considered problems can be solved exactly in the case of polytopic uncertainty description. We remind that given a finite dimension vector space $\mathcal{V}$ we call a polytope with vertices $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s \in \mathcal{V}$ the set $\mathcal{P} \subset \mathcal{V}$ defined as follows:

$$\mathcal{P} = \left\{ \mathbf{v} = \sum_{i=1}^{s} p_i \mathbf{v}_i, \sum_{i=1}^{s} p_i = 1, p_i \geq 0 \right\}.$$

We call a polyhedron a set defined by linear equalities and inequalities:

$$\mathcal{P} = \{\mathbf{v} : \mathbf{d}_k^{\mathrm{T}}\mathbf{v} = e_k,\ k = 1, 2, \ldots, \bar{k}, \quad \mathbf{f}_i^{\mathrm{T}}\mathbf{v} \leq g_i,\ i = 1, 2, \ldots, \bar{i}\}. \tag{5}$$

Any bounded polyhedron is a polytope and vice versa (see [16]) so the two representations above are equivalent in the case of compact polyhedral sets. Accordingly, we introduce the following definition.

Definition 2.1. The system has a *polytopic uncertainty description* if

$$\mathbf{M}(\mathbf{p}) = \sum_{i=1}^{s} \mathbf{M}_i p_i, \tag{6}$$

$$\mathbf{K}(\mathbf{p}) = \sum_{i=1}^{s} \mathbf{K}_i p_i \tag{7}$$

and the vector $\mathbf{p}$ belongs to an assigned polytope $\mathcal{P}$. We say that the system is in the standard representation if $\mathcal{P}$ is a simplex, namely

$$\sum_{i=1}^{s} p_i = 1, \quad p_i \geq 0. \tag{8}$$

The symmetric matrices $\mathbf{M}_i$ and $\mathbf{K}_i$ are called the vertex matrices or generating matrices.

We point out that the description (6) and (7) is not in general the "natural one", namely, the one which is assigned in practice since, in general, the admissible region is assigned by means of linear equalities and inequalities, not necessarily as in Definition 2.1. However, the following essential lemma holds [16].

Lemma 2.1. *Any system of the form* (1) *in which* $[\mathbf{M}(\mathbf{p}), \mathbf{K}(\mathbf{p})]$ *are of the form* (6) and (7) *and in which the parameter vector* $\mathbf{p} \in \mathcal{P}$, *where* $\mathcal{P}$ *is defined by the constraint* (5), *can be equivalently represented as in* (6)–(8). *Precisely the matrix family always admits a suitable parametrization in terms of a vector* $\hat{\mathbf{p}}$ *which satisfies* (8).

Note that if $\mathbf{M}$ and $\mathbf{K}$ are functions of different parameters $\mathbf{p}^M$ and $\mathbf{p}^K$, this is a special case since we can always form an augmented parameter vector

$$\mathbf{p} = \begin{bmatrix} \mathbf{p}^M \\ \mathbf{p}^K \end{bmatrix},$$

which is linearly constrained as long as its components $\mathbf{p}^M$ and $\mathbf{p}^K$ are such.

A special case is that in which $\mathbf{p}$ lies in a box as follows:

$$\overline{p}_i^- \leq p_i \leq \overline{p}_i^+, \quad i = 1, 2, \ldots, s.$$

In this case, we can consider $\hat{s} = 2^s$ generating matrices

$$\hat{\mathbf{M}}_h = \sum_{i=1}^{s} \mathbf{M}_i \overline{\mathbf{p}}_i^h, \quad \hat{\mathbf{K}}_h = \sum_{i=1}^{s} \mathbf{K}_i \overline{\mathbf{p}}_i^h,$$

where the $\overline{\mathbf{p}}_i^h$ are taken on the extrema of the intervals

$$\overline{\mathbf{p}}_i^h \in \{\overline{p}_i^-, \overline{p}_i^+\}.$$

For instance if $p_1^- \leq p_1 \leq p_1^+$ and $p_2^- \leq p_2 \leq p_2^+$ the four vertex matrices are

$$\hat{\mathbf{M}}_1 = \mathbf{M}_1 p_1^- + \mathbf{M}_2 p_2^-, \quad \hat{\mathbf{M}}_1 = \mathbf{M}_1 p_1^- + \mathbf{M}_2 p_2^+, \quad \hat{\mathbf{M}}_3 = \mathbf{M}_1 p_1^+ + \mathbf{M}_2 p_2^-, \quad \hat{\mathbf{M}}_4 = \mathbf{M}_1 p_1^+ + \mathbf{M}_2 p_2^+$$

and the family of matrices can be equivalently represented as

$$\mathbf{M} = \sum_{i=1}^{4} \hat{\mathbf{M}}_i \hat{p}_i \ \text{ with } \ \sum_{i=1}^{4} \hat{p}_i = 1 \quad \text{and} \quad \hat{p}_i \geq 0$$

(the same expression obviously holds for the $\mathbf{K}$ matrix). Finally note that the considered structure includes as special case the interval systems, namely systems whose uncertainty bounds are given componentwise considered in [3]:

$$\mathbf{M}_{ik}^- \leq \mathbf{M}_{ik} \leq \mathbf{M}_{ik}^+, \quad \mathbf{K}_{ik}^- \leq \mathbf{K}_{ik} \leq \mathbf{K}_{ik}^+.$$

Remark 2.1. The parametrization (6)–(8) is adopted for theoretical purposes only, but not used in practice. We will see that, for computational purposes, we just need to compute the vertices of the assigned polyhedron $\mathcal{P}$ by means of well established linear programming tools and to analyze the systems corresponding to the vertices. The reader is referred to the next subsection to have an example of how the "vertex" systems can be derived starting from the "natural" representation in terms of inequalities.

2.2. Motivating example

Consider the structure in Fig. 1. Here we assume that both the lengths and the masses are unknown. For instance the masses may be unknown because the structure additional load is uncertain. The column length can be uncertain because the structure is re-configurable or it is under design. If we neglect axial deformations of all the beams and we take horizontal displacements of the floors as Lagrangian variables, then the mass matrix is diagonal

$$\mathbf{M}(m_1, m_2, m_3, m_4) = \mathrm{DIAG}\{m_1, m_2, m_3, m_4\}.$$

The stiffness matrix turns out to be

$$\begin{bmatrix} k_1 + k_2 & -k_2 & 0 & 0 \\ -k_2 & k_2 + k_3 & -k_3 & 0 \\ 0 & -k_3 & k_3 + k_4 & -k_4 \\ 0 & 0 & -k_4 & k_4 + k_d \end{bmatrix}.$$

The situation we wish to analyze is that in which there are only bounds available imposed either by the knowledge of the system or by design specification. The easier way to impose specifications is to assume proper admissibility intervals such as

$$m_i^- \le m_i \le m_i^+ \tag{9}$$

and

$$k_i^- \le k_i \le k_i^+. \tag{10}$$

Note that, in this case, while $\mathbf{M}$ is an interval matrix, $\mathbf{K}$ is not such, since the coefficients are correlated. In this simple situation, physical considerations lead to the obvious conclusion that all the frequency intervals have as lower (respectively, upper) extrema the frequencies which correspond to $\mathbf{M}_i^+$ and h_i^- (respectively, $\mathbf{M}_i^-$ and h_i^+). This special case will be discussed later on.

Clearly this situation is not always the actual one. For instance in most cases the overall mass is known but it is not known how the mass is distributed on the floors. This introduces a constraint of the form

$$m_i = \overline{m}_i + \mu_i,$$

$$\sum_{i=1}^{4} \mu_i = \overline{\mu},$$

$$\mu_i \ge 0,$$

where $\overline{m}_i$ is the mass of the empty ith floor and μ_i is the portion of the additional mass assigned to the ith floor. Note that this kind of uncertainty cannot be faced using an "interval analysis" approach. The vertices of the overall system $[\mathbf{M}, \mathbf{K}]$ are

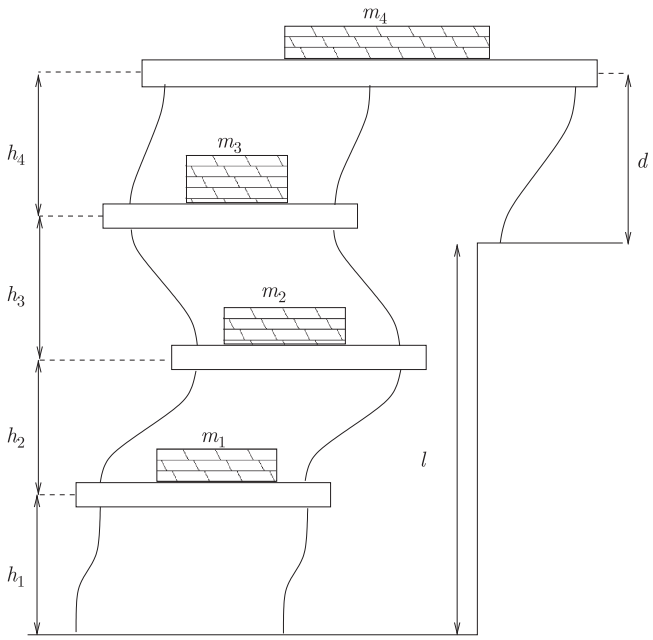

Fig. 1. Flexible structure with uncertain parameters.

achieved by assuming one of the μ_i's equal to $\overline{\mu}$ and the remaining ones equal to zero, and by assuming the k_i's on their extrema in all possible ways. For instance $\mu_1=\mu_3=\mu_4=0$, $\mu_2=\overline{\mu}$, $k_1=k_1^+$, $k_2=k_2^+$, $k_3=k_3^-$, $k_4=k_4^+$, corresponds to a vertex of the system. Therefore, this system has $4\times 2^4=64$ vertices.

If the lengths h_i are uncertain then we can model the situation by considering nominal lengths $\overline{h}_i$ and uncertain variations v_i for each of them. If we assume that only the overall length is given then we have to consider the following constraints.

$$h_i=\overline{h}_i+v_i,$$

$$\sum_{i=1}^{4} v_i=0,$$

$$v_i^-\le v_i^+.$$

Note that, however, the situation is different from that of the uncertain mass, because the stiffness parameters k_i are typically of the form

$$k_i=\frac{\overline{k}_i}{h_i^3}=\frac{\overline{k}_i}{(\overline{h}_i+v_i)^3}$$

thus depend in a nonlinear way on the uncertain parameters v_i. If we assume that the overall uncertainty is small, denoting by $h_i=\overline{h}_i+\delta_i$ we have

$$h_i\approx\frac{\overline{k}_i}{\overline{h}_i^3}-3\frac{\overline{k}_i}{\overline{h}_i^4}\delta_i$$

then the linear analysis we are considering is still effective. Assuming "small perturbations" is reasonable in many situations and other approaches (for instance [1]) require this assumption. Possible extensions to some classes of models with nonlinear uncertainties will be discussed later.

3. Main results

In this section we investigate the "extreme point property" according to the following definition.

Definition 3.1. We say that a certain set $\mathcal{P}$ has the strong (weak) extreme-point property if the strong (weak) root interval detection problem can be solved by computing the roots of all the systems achieved by taking $p\in vert\,\mathcal{P}$, the set of vertices of $\mathcal{P}$.

The first result of the section is the next theorem.

Theorem 3.1. *Any system with polytopic uncertainty set has the weak extreme-point property.*

Proof. Let us consider the Rayleigh's quotient

$$\rho(\mathbf{p},\mathbf{q})=\frac{\mathbf{q}^{\mathrm{T}}\mathbf{K}(\mathbf{p})\mathbf{q}}{\mathbf{q}^{\mathrm{T}}\mathbf{M}(\mathbf{p})\mathbf{q}}.$$

It is known that the maximum eigenvalue can be expressed as

$$\omega_m^2(\mathbf{p})=\max_{\|\mathbf{q}\|=1}\frac{\mathbf{q}^{\mathrm{T}}\mathbf{K}(\mathbf{p})\mathbf{q}}{\mathbf{q}^{\mathrm{T}}\mathbf{M}(\mathbf{p})\mathbf{q}}.$$

Let us now consider, for the moment being, $\|\mathbf{q}\|=1$ be fixed. Take any pair of points $\mathbf{p}^A$ and $\mathbf{p}^B$ inside $\mathcal{P}$ and take their convex combination

$$\mathbf{p}=\alpha\mathbf{p}^A+(1-\alpha)\mathbf{p}^B$$

$0\le\alpha\le 1$. Define the function

$$\varphi(\alpha)\doteq\rho(\alpha\mathbf{p}^A+(1-\alpha)\mathbf{p}^B,\mathbf{q})=\frac{\alpha\mathbf{q}^{\mathrm{T}}\mathbf{K}(\mathbf{p}^A)\mathbf{q}+(1-\alpha)\mathbf{q}^{\mathrm{T}}\mathbf{K}(\mathbf{p}^B)\mathbf{q}}{\alpha\mathbf{q}^{\mathrm{T}}\mathbf{M}(\mathbf{p}^A)\mathbf{q}+(1-\alpha)\mathbf{q}^{\mathrm{T}}\mathbf{M}(\mathbf{p}^B)\mathbf{q}}.$$

In view of the linearity of $\mathbf{K}$ and $\mathbf{M}$, $\varphi(\alpha)$ is a bi-linear function of α. By assumption, $\mathbf{M}(\mathbf{p})$ is positive definite and thus the denominator is different from zero for all α. The maximum of $\varphi(\alpha)$ is reached on the extrema (either $\alpha=0$ or $\alpha=1$) since, as it is easy to check, $\mathrm{d}\varphi(\alpha)/\mathrm{d}\alpha$ does not change sign on $[0,1]$. Then

$$\max_{0\le\alpha\le 1}\rho(\alpha\mathbf{p}^A+(1-\alpha)\mathbf{p}^B,\mathbf{q})=\frac{\alpha\mathbf{q}^{\mathrm{T}}\mathbf{K}(\mathbf{p}^A)\mathbf{q}+(1-\alpha)\mathbf{q}^{\mathrm{T}}\mathbf{K}(\mathbf{p}^B)\mathbf{q}}{\alpha\mathbf{q}^{\mathrm{T}}\mathbf{M}(\mathbf{p}^A)\mathbf{q}+(1-\alpha)\mathbf{q}^{\mathrm{T}}\mathbf{M}(\mathbf{p}^B)\mathbf{q}}=\max\{\rho(\mathbf{p}^A,\mathbf{q}),\rho(\mathbf{p}^B,\mathbf{q})\}.$$

The next step of the proof is to show that, necessarily, for fixed $\|\mathbf{q}\|=1$, $\rho(\mathbf{p})$ reaches its maximum on the boundary. This is immediate, since for any point $\mathbf{p}$ in the interior of $\mathcal{P}$ we can take two points p_1 and p_2 on the boundary of $\mathcal{P}$ such that $\mathbf{p}$ is in

their convex combination thus necessarily $\rho(\mathbf{p},\mathbf{q}) \le \max\{\rho(\mathbf{p}^A,\mathbf{q}), \rho(\mathbf{p}^B,\mathbf{q})\}$, as we have just seen. We conclude that $\mathbf{P}$ must be necessarily on a face of the polytope $\mathcal{P}$. On the other hand, any face of a polytope is also a polytope [16]. Let us consider the hyperplane including such a face. By the same reasoning we can conclude that $\rho(\mathbf{p},\mathbf{q})$ cannot reach the maximum on the relative interior of the facet. By iterating these considerations we have necessarily that $\rho(\mathbf{p},\mathbf{q})$ reaches its maximum on a vertex.

Then,denote by $\hat{\mathbf{v}}_i$ the vertices[1] of $\mathcal{P}$ and $\mathbf{M}(\hat{\mathbf{v}}_i) = \mathbf{M}_i$. Denote by $\omega_m^2(\hat{\mathbf{v}}_i)$ the largest eigenvalue of the pair $(\mathbf{M}(\hat{\mathbf{v}}_i), \mathbf{K}(\hat{\mathbf{v}}_i))$. Then

$$\omega_m^2(\mathbf{p}) = \max_{p\in\mathcal{P}} \max_{\|\mathbf{q}\|=1} \frac{\mathbf{q}^T\mathbf{K}(\mathbf{p})\mathbf{q}}{\mathbf{q}^T\mathbf{M}(\mathbf{p})\mathbf{q}} = \max_{\|\mathbf{q}\|=1} \max_{p\in\mathcal{P}} \frac{\mathbf{q}^T\mathbf{K}(\mathbf{p})\mathbf{q}}{\mathbf{q}^T\mathbf{M}(\mathbf{p})\mathbf{q}} = \max_{\|\mathbf{q}\|=1} \max_{p\in vert\{\mathcal{P}\}} \frac{\mathbf{q}^T\mathbf{K}(\mathbf{p})\mathbf{q}}{\mathbf{q}^T\mathbf{M}(\mathbf{p})\mathbf{q}} = \max_{\|\mathbf{q}\|=1} \max_i \frac{\mathbf{q}^T\mathbf{K}(\hat{\mathbf{v}}_i)\mathbf{q}}{\mathbf{q}^T\mathbf{M}(\hat{\mathbf{v}}_i)\mathbf{q}}$$
$$= \max_i \max_{\|\mathbf{q}\|=1} \frac{\mathbf{q}^T\mathbf{K}(\hat{\mathbf{v}}_i)\mathbf{q}}{\mathbf{q}^T\mathbf{M}(\hat{\mathbf{v}}_i)\mathbf{q}} = \max_i \omega_m^2(\hat{\mathbf{v}}_i).$$

The first two equalities are obvious. The third and the fourth hold in view of what has been previously proved. The fifth is again obvious and the sixth holds by the definition of $\omega_m^2(\hat{\mathbf{v}}_i)$.

The proof for the "minimum eigenvalue" is now straightforward. Just note that we have to consider two cases:

- $\mathbf{K}(\mathbf{p})$ is only positive semi-definite. Then the minimum eigenvalue is 0. In this case we have to note that necessarily one of the vertices $\mathbf{K}_i$ is singular namely $\mathbf{q}^T\mathbf{K}_i\mathbf{q} = 0$ for some $\|\mathbf{q}\| = 1$. Indeed if it would hold $\mathbf{q}^T\mathbf{K}_i\mathbf{q} > 0$ for all $\|\mathbf{q}\| = 1$ then for $\mathbf{p}$ such that $\sum_i p_i = 1$ and $p_i \ge 0$
$$\mathbf{q}^T\mathbf{K}(\mathbf{p})\mathbf{q} = \sum_i p_i[\mathbf{q}^T\mathbf{K}_i\mathbf{q}] > 0.$$
Therefore the case in which the smallest eigenvalue is zero is easily detected.
- $\mathbf{K}(\mathbf{p})$ is positive definite for all $\mathbf{p}$. Then the proof follows immediately by replacing "max" by "min". □

Corollary 3.1. *If the system has a polytopic uncertainty set, then the largest and the smallest natural frequencies are achieved by the vertex systems $\mathbf{M}_i, \mathbf{K}_i$ so that they can be evaluated by considering a finite number of eigenvalue problems as*

$$a = \min_i \ \{\textit{eigenvalues of } \lambda\mathbf{M}_i - \mathbf{K}_i\}$$

and

$$b = \max_i \ \{\textit{eigenvalues of } \lambda\mathbf{M}_i - \mathbf{K}_i\}.$$

In the special case of "box type" bounding regions

$$\mathbf{M}(\mathbf{p}) = \mathbf{M}_0 + \sum_{i=1}^{s} \mathbf{M}_i p_i^{\mathbf{M}}, \quad \mathbf{K}(\mathbf{p}) = \mathbf{K}_0 + \sum_{i=1}^{s} \mathbf{K}_i \mathbf{p}_i^K,$$

with

$$\overline{p}_i^{M-} \le \mathbf{p}_i^M \le \overline{p}_i^{M+}, \quad \overline{p}_i^{K-} \le \mathbf{p}_i^K \le \overline{p}_i^{K+}$$

the following theorem holds. Basically, this is the same result previously presented in [11], reported here for the sake of completeness.

Theorem 3.2. *If the matrices $\mathbf{K}_i$ and $\mathbf{M}_i$ are all positive semi-definite, then the strong root interval detection can be solved by computing the eigenvalues of two pair of matrices. Precisely consider the extrema of the intervals $[a_i, b_i]$ as defined in Eqs.* (3)–(4)*, then*:

- *a_i are the eigenvalues of $\lambda[\mathbf{M}_0 + \sum_{i=1}^{s} \mathbf{M}_i\overline{p}_i^{M+}] - [\mathbf{K}_0 + \sum_{i=1}^{s} \mathbf{K}_i\overline{p}_i^{K-}]$;*
- *b_i are the eigenvalues of $\lambda[\mathbf{M}_0 + \sum_{i=1}^{+} \mathbf{M}_i\overline{p}_i^{M-}] - [\mathbf{K}_0 + \sum_{i=1}^{s} \mathbf{K}_i\overline{p}_i^{K+}]$.*

Proof. Take a symmetric matrix $\mathbf{S}$, and denote by σ_i its eigenvalues ordered by magnitude $\sigma_1 \le \sigma_2 \le \cdots \le \sigma_m$. Take a positive semi-definite matrix $\mathbf{M}$ and let $\tilde{\mathbf{S}} = \mathbf{S} + \mathbf{M}$ with ordered eigenvalues $\tilde{\sigma}_1 \le \tilde{\sigma}_2 \le \cdots \le \tilde{\sigma}_m$. Then

$$\tilde{\sigma}_i \ge \sigma_i \quad \text{for all } i.$$

Then if we replace any parameter p_i by its upper bound $\overline{p}_i^+$ we achieve a set of eigenvalues which are all increased (or not decreased). Therefore the maximum of all frequencies is necessary achieved corresponding to the maximum values of the parameters. □

[1] Note that if we consider the standard representation (6)–(8) we just have $\hat{\mathbf{v}}_i = [0\ 0\ 0\ \ldots\ 0\ 1\ 0\ \ldots\ 0]$, $\mathbf{K}(\hat{\mathbf{v}}_i) = \mathbf{K}_i$.

We stress that, in general, the detected intervals may be overlapping and only in very special cases we can assure that this phenomenon does not occur. The interested reader is referred to [17, Section 5.3.2, item (5)].

Remark 3.1. In general the interval systems do not have the strong extreme point property, unless we consider additional assumptions as in [4]. For instance assume $\mathbf{M}=\mathbf{I}$ and

$$\mathbf{K}=\begin{bmatrix} 3 & -b \\ -b & 3 \end{bmatrix}$$

$-1\le b\le 1$. The two frequency intervals are $\lambda_{\min}\in[2,3]$ and $\lambda_{\max}\in[3,4]$. The internal value 3 common to both interval is not achieved on the extrema ± 1 but for $b=0$, namely in the interior of the interval. Interestingly enough, if we abandon the symmetry then the strong interval detection can be solved [12], as long as we can assume that all the eigenvalues of $\hat{\mathbf{K}}$ are real. For instance, take

$$\hat{\mathbf{K}}=\begin{bmatrix} 3 & -b_2 \\ -b_1 & 3 \end{bmatrix},$$

with $1\le b_i\le 2$, then the two intervals are $\lambda_{\min}\in[1,2]$ and $\lambda_{\max}\in[4,5]$ and their extrema are on the vertices in agreement with previous results [4,12].

4. More general uncertainty structures and embedding

A class of systems, which generalizes the one previously considered, having the weak extreme-point property is introduced next.

Definition 4.1. The uncertainty has a *multi-linear fractional description* if

$$\mathbf{M}(\mathbf{p})=\frac{\mathbf{M}_0+\sum \mathbf{M}_{i_1,i_2,\dots,i_r}p_1^{i_1}p_2^{i_2}\dots p_r^{i_r}}{v_0+\sum \mu_{i_1,i_2,\dots,i_r}p_1^{i_1}p_2^{i_2}\dots p_r^{i_r}}, \tag{11}$$

$$\mathbf{K}(\mathbf{p})=\frac{\mathbf{K}_0+\sum \mathbf{K}_{i_1,i_2,\dots,i_r}p_1^{i_1}p_2^{i_2}\dots p_r^{i_r}}{v_0+\sum \mu_{i_1,i_2,\dots,i_r}p_1^{i_1}p_2^{i_2}\dots p_r^{i_r}}, \tag{12}$$

$$p_i^-\le p_i\le p_i^+, \tag{13}$$

where $i_k\in\{0,1\}$, the matrices $\mathbf{M}_{i_1,i_2,\dots,i_r}$, $\mathbf{K}_{i_1,i_2,\dots,i_r}$, the scalars $\mu_{i_1,i_2,\dots,i_r}$ and v_0 are given.

As an example, consider the case where $\mathbf{M}$ is diagonal and $\mathbf{K}$ depends linearly on the parameters. If we divide by $\mathbf{M}$, we get the model $\ddot{\mathbf{q}}=-\mathbf{M}^{-1}\mathbf{K}\mathbf{q}$, and matrix $\mathbf{M}^{-1}\mathbf{K}$ has a multi-linear fractional description. Take for instance the model of Section 2.2 to derive the multi-linear fractional description

$$\mathbf{M}^{-1}\mathbf{K}=\frac{\begin{bmatrix} M_2M_3M_4(k_1+k_2) & -M_2M_3M_4k_2 & 0 & 0 \\ -M_1M_3M_4k_2 & M_1M_3M_4(k_2+k_3) & -M_1M_3M_4k_3 & 0 \\ 0 & -M_1M_2M_4k_3 & M_1M_2M_4(k_3+k_4) & -M_1M_2M_4k_4 \\ 0 & 0 & -M_1M_2M_3k_4 & M_1M_2M_3(k_4+k_d) \end{bmatrix}}{M_1M_2M_3M_4}.$$

We define "vertex matrices" all the matrices we can achieve by taking p_i on the extrema

$$\{\hat{\mathbf{M}}_i=\mathbf{M}(\hat{p}),\hat{p}_i\in\{p_i^-,p_i^+\}\},\quad \{\hat{\mathbf{K}}_i=\mathbf{K}(\hat{p}),\hat{p}_i\in\{p_i^-,p_i^+\}\}.$$

Theorem 4.1. *Any system with fractional uncertainty set has the weak extreme-point property.*

Proof. The following properties are required for the proof.

- Given the family of symmetric matrices $\mathbf{M}(\mathbf{p})$ with bi-linear structure, $\mathbf{M}(\mathbf{p})$ is positive definite for all $\mathbf{p}$ if and only if all the "vertex matrices" $\hat{\mathbf{M}}_i$ are all positive definite.
- Given a symmetric matrix $\mathbf{S}$ its largest eigenvalue $\overline{\sigma}$ is

$$\overline{\sigma}=\inf\{\sigma:(\mathbf{S}-\sigma\mathbf{I}) \text{ is negative definite}\}.$$

The proof of the first claim is reported in [18]. The second claim is immediate since the eigenvalues of $\mathbf{S}-\sigma\mathbf{I}$ are those of $\mathbf{S}$ translated by $-\sigma$, and $\mathbf{S}-\sigma\mathbf{I}$ is negative definite if and only if all its eigenvalues are negative.

The proof now requires the following extension. Given a symmetric matrix $\mathbf{S}$ and a positive definite matrix $\mathbf{R}$ the largest eigenvalue $\overline{\sigma}$ of $\sigma\mathbf{R}-\mathbf{S}$ has the same property indeed $(\mathbf{S}-\sigma\mathbf{R})$ is definite negative iff $\mathbf{R}^{-1/2}(\mathbf{S}-\sigma\mathbf{R})\mathbf{R}^{-1/2}=\mathbf{R}^{-1/2}\mathbf{S}\mathbf{R}^{-1/2}-\sigma\mathbf{I}$ is such. Then

$$\begin{aligned}\overline{\sigma} &= \inf\{\sigma : (\mathbf{S}-\sigma\mathbf{R}) \text{ is negative definite}\} = \inf\{\sigma : (\mathbf{R}^{-1/2}\mathbf{S}\mathbf{R}^{-1/2}-\sigma\mathbf{I}) \text{ is negative definite}\}\\ &= \max\{\text{eigenvalues of } (\mathbf{R}^{-1/2}\mathbf{S}\mathbf{R}^{-1/2}-\sigma\mathbf{I})\} = \max\{\text{eigenvalues of } (\mathbf{S}-\sigma\mathbf{R})\}.\end{aligned}$$

The proof is completed if we put together the previous statements. Indeed, for any σ, $\mathbf{M}(\mathbf{p})\sigma-\mathbf{K}(\mathbf{p})$ is negative definite for all $\mathbf{p}$ if and only if the vertices $\hat{\mathbf{M}}_i\sigma-\hat{\mathbf{K}}_i$ are negative definite for all i. Thus the largest eigenvalue of $\mathbf{M}(\mathbf{p})\sigma-\mathbf{K}(\mathbf{p})$ is maximum of the dominant eigenvalues of $\hat{\mathbf{M}}_i\sigma-\hat{\mathbf{K}}_i$.

The case of the smallest eigenvalue is identical. □

4.1. Embedding

There are cases in which the uncertainty bounding set is not a polyhedron and thus it is not described via linear equalities or in which the entries of $\mathbf{M}$ and $\mathbf{K}$ are not affine functions of the uncertain parameters. Then, the extreme point property does not hold in general. Consider the single degree of freedom system in Fig. 2 representing a shaft–flywheel body. Assume that the total length L is fixed but the portion d of it attributed to the flywheel and the portion of the shaft l are not determined. So assuming l as parameter so that $d=L-l$ we get $J(l)=J_0(L-l)$ and $k(l)=k_0/l$. Assume $l^-\le l\le l^+$. The resulting frequency is

$$\omega=\sqrt{\frac{k_0}{J_0(L-l)l}},$$

which is a convex differentiable function. It is clear to see that in general the maximum value of such a frequency is not on the extrema of such an interval. Therefore the general case is much more complex.

Given a system $[\mathbf{M}(\mathbf{p}),\mathbf{K}(\mathbf{p})]$ and $\mathbf{p}\in\mathcal{P}$, one can consider the minimization (maximization) problem

$$\min\ \omega_1(\mathbf{p}),\ \ \mathbf{p}\in\mathcal{P}\quad\text{or}\quad\max\ \omega_m(\mathbf{p}),\ \ \mathbf{p}\in\mathcal{P}.$$

Clearly in general there is no guarantee of convergence, and the solver can be trapped in local minima. As an alternative approach we can embed $\mathbf{M}(\mathbf{p}),\mathbf{K}(\mathbf{p})$ in a larger family, namely

$$\mathbf{M}(\mathbf{p}),\mathbf{K}(\mathbf{p})=\mathbf{M}(\tilde{\mathbf{p}}),\mathbf{K}(\tilde{\mathbf{p}})\quad\text{for some }\tilde{\mathbf{p}}\in\tilde{\mathcal{P}},$$

with a "nice structure". Then by applying our results to the larger family, we achieve lower and upper bounds for the overall frequency interval.

Note that this procedure is not alternative but complementary to the "optimization" procedure. In general, if we seek for the minimum $\omega_1(\mathbf{p})$, then the optimization solver gives an upper bound, while the absorbing procedure provides a lower bound. The opposite is true for the maximum eigenvalue $\omega_m(\mathbf{p})$.

To show an example of an absorbing procedure, consider the structure in Fig. 1. If we take the horizontal displacements as variables, the mass matrix is diagonal. Now we remind that

$$k_i=\frac{\overline{k}_i}{(\overline{h}_i+v_i)^3},$$

with $\sum v_i=0$. Assume that $v_i\le\overline{v}_i$ is the given bound. Then we have

$$\frac{\overline{k}_i}{(\overline{h}_i+\overline{v}_i)^3}\le k_i\le\frac{\overline{k}_i}{(\overline{h}_i-\overline{v}_i)^3}. \tag{14}$$

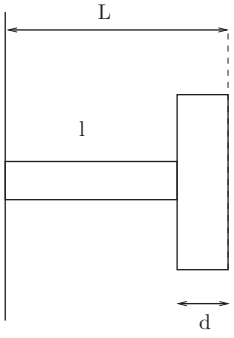

Fig. 2. The shaft–flywheel system.

This is a linear constraint. However the constrain $\sum v_i = 0$ gives troubles. We have

$$v_i = -\overline{h}_i + \sqrt[3]{\frac{\overline{k}_i}{k_i}}$$

then

$$\sum \sqrt[3]{\frac{\overline{k}_i}{k_i}} = \sum \overline{h}_i.$$

Thus we get a nonlinear constraint on the k_i on the interval (14). Let us now relax this equality to an inequality

$$\sum \sqrt[3]{\frac{\overline{k}_i}{k_i}} \leq \sum \overline{h}_i$$

along with the intervals (14). Then we can take any linear function $\alpha_i k_i + \beta_i$ which is dominated by this function namely such that

$$\alpha_i k_i + \beta_i \leq \sqrt[3]{\frac{\overline{k}_i}{k_i}} \leq \sum \overline{h}_i$$

for all i. Therefore we derive a set of linear constraints. Clearly the old (true) feasible set for the parameters is a subset of the new one, and thus the intervals we derive are larger, in general, than the actual ones.

5. Minimizing (maximizing) the maximum (minimum) frequency

We consider the case in which the parameters are not uncertain but they are decision variables. This basically means that $\mathbf{p} \in \mathcal{P}$ is not intended as "$\mathbf{p}$ is determined by circumstances" but it is a free design parameter. Then the following goals may be of interest:

- maximizing the smallest frequency;
- minimizing the largest frequency.

These problems may be of interest in those cases in which the structure has to be protected from a noise whose spectrum is dominated by a frequency which should not collide with natural frequencies. Let us consider the problem of rendering the structure sub-critical (the other problem is exactly the same) and, accordingly, we wish to compute

$$\min_{p \in \mathcal{P}} \omega_m(\mathbf{p}).$$

This type of problem minimizing the largest eigenvalue is known in the literature [19]. For this problem a vertex result does not hold in general in the sense that the maximum frequency is minimized in a point in $\mathcal{P}$ which is not necessarily on a vertex. This is the case, for instance, of the system

$$\mathbf{M} = \begin{bmatrix} 1+2p_1 & 0 \\ 0 & 1+2p_2 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix},$$

with $p_1 > 0$, $p_2 > 0$ and $p_1 + p_2 = 1$. It is immediate that the low and high frequencies ω_1 and ω_2 are, respectively, minimized and maximized on either of the extrema $p_1 = 1$ and $p_2 = 0$ (or the opposite) precisely we get $\omega_1^2 = 0.45142$ and $\omega_2^2 = 2.21525$. Conversely ω_1 and ω_2 are maximized and minimized in the middle $p_1 = p_2 = \frac{1}{2}$ where $\omega_1^2 = 0.5$ and $\omega_2^2 = 1.5$.

The problem has the property that as long as $\mathcal{P}$ is a convex set, it is a convex optimization problem for which efficient algorithms are available for its solution. In particular, the convexity of the problem assures that any local minimum is a global one, therefore any gradient-type optimization algorithm assures convergence to the optimal.

6. Application to the uncertain mass distribution case

As an application of the proposed result, we consider the problem arising in structures in which the load is not known. In particular we consider the case in which the mass is given but its spatial distribution is uncertain. Note that the case in which the total amount of mass is also uncertain and $\mathbf{K}$ is fixed, is trivial. Indeed, according to Theorem 3.2 (and obvious physical considerations), given a mass partition, all the natural frequencies are decreasing function of the total mass, therefore variations of the total mass, within a given interval, can be easily dealt with by repeating our "uncertain distribution analysis" twice by considering the minimum and maximum mass.

6.1. Example 1

Consider the two-story frame presented in Fig. 3. Both floors are assumed to be rigid and composed of three square fields ($4 \times 4\,m^2$). Mass density is assumed to be constant and equal to $250\,kg/m^2$. All columns have the same flexural stiffness and are assumed axially rigid. This structure has 6 degrees of freedom, namely, two translational and one rotational for each floor. The Lagrangian coordinate vector is then $[x_1\ y_1\ \theta_1\ x_2\ y_2\ \theta_2]$ where (x_1, y_1) and (x_2, y_2) are the positions of the reference points of the first and the second floor, respectively, while θ_1 and θ_2 are the rotation of the first and the second floor.

We assume that a certain amount of a total mass $\overline{\mu}$ can be arbitrarily distributed among the 16 points located at the vertices of the square field. Denoting by μ_i, $i = 1, 2, \ldots, 16$, the portion of mass placed on the ith point, the equality constraint $\sum_i \mu_i = \overline{\mu}$, $\mu_i \geq 0$ must hold. The situation corresponds to that presented in Section 2.2. According to that theory, to determine the minimum and the maximum frequency, we need to examine 16 cases, namely $\mu_i = \overline{\mu}$, $i = 1, \ldots, 16$. Since the structure is symmetric, we actually consider only the points numbered in Fig. 4.

This example shows that the position of the mass which yields the minimum/maximum natural frequencies is not a priori obvious. Indeed, one would expect that the maximum frequency is reached by placing the additive mass on the position 5 on the first floor (see Fig. 4 for the numeration of the nodes), closest as possible to the center of mass and to the center of stiffness. Instead, the maximum frequency (9.689 Hz) is reached placing the additive mass on position 10 on the second floor (bright ball on Fig. 3). The minimum frequency is reached by concentrating the mass on point 6 (2.377 Hz). Thus the overall interval turns out to be

$$[2.377, 9.689].$$

We stress that this interval is the largest interval *for all possible distribution of the additive mass μ on the points*. The first natural frequency and the last (sixth) frequency achieved by placing all the mass in the considered points are reported in Tables 1 and 2 corresponding to $\overline{\mu} = 2000\,kg$.

It is interesting to observe that the conclusion reached for the minimum natural frequency holds only for small values of $\overline{\mu}$, since if $\overline{\mu}$ is greater than about 3200 kg, then the minimum natural frequency is reached by placing all the mass on the point 8 (Fig. 5).

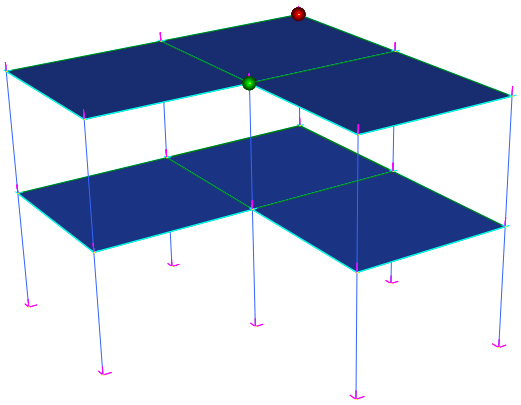

Fig. 3. Two-story frame.

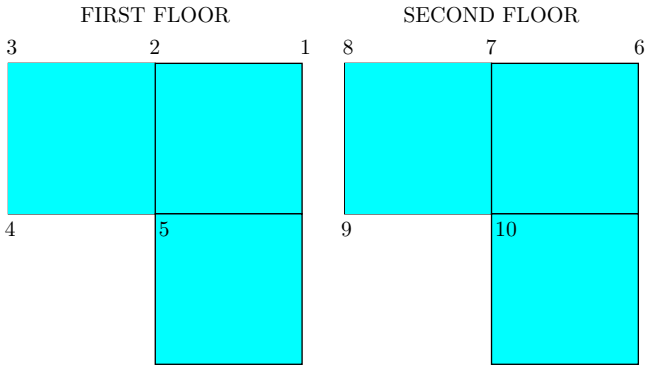

Fig. 4. Numeration of the points in model 2.

Table 1
Minimum natural frequency.

Position	4	5	3	2	1	10	9	8	7	6
Frequency	2.529	2.527	2.516	2.516	2.504	2.444	2.429	2.390	2.390	2.377

Table 2
Maximum natural frequency.

Position	10	5	6	9	8	7	2	1	4	3
Frequency	9.689	9.652	9.434	9.378	9.298	9.298	9.284	9.006	8.842	8.636

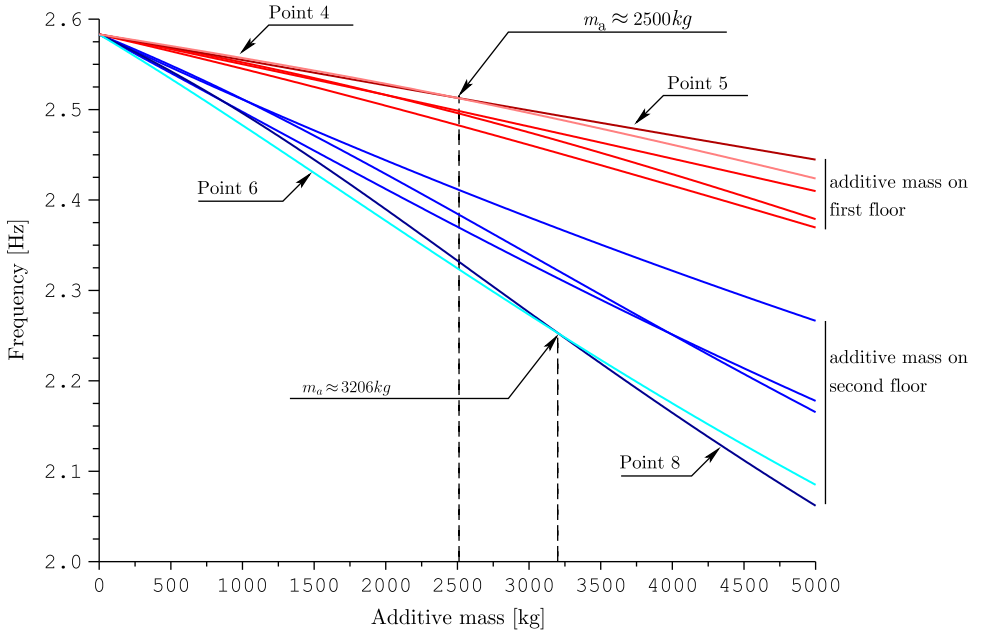

Fig. 5. Minimum frequencies versus additive mass on different points.

The structure has been further investigated in order to consider the case in which the mass is not necessarily concentrated on the specified points but it can be distributed on the floors. By adding a unit mass at a generic point of coordinates $[b_x,\ b_y]$ (with respect to the reference point) the mass matrix is increased with the term

$$\Delta\mathbf{M}_1 = \begin{bmatrix} 1 & 0 & -b_y & 0 & 0 & 0 \\ 0 & 1 & b_x & 0 & 0 & 0 \\ -b_y & b_x & b_x^2 + b_y^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \cdot 1$$

if the mass is on the first floor or

$$\Delta\mathbf{M}_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -b_y \\ 0 & 0 & 0 & 0 & 1 & b_x \\ 0 & 0 & 0 & -b_y & -b_x & b_x^2 + b_y^2 \end{bmatrix} \cdot 1$$

if it is on the second floor. If additive masses $m_{a,1}$ and $m_{a,2}$ are placed both at the first and the second floor, we can write

$$\mathbf{M} = \mathbf{M}_0 + \Delta\mathbf{M_1} m_{a,1} + \Delta\mathbf{M_2} m_{a,2},$$

where $\mathbf{M}_0$ is the unloaded mass matrix. Although we do not have a formal theory for a continuous mass distribution, the lowest and the highest frequencies are achieved by concentrating the mass in a single point on the floor (the extremal mass distribution is a Dirac function). To render the argument solid we can reason as follows. If we take a grid of arbitrarily many points on the floors, and if we distribute the mass among them, we achieve a mass matrix which is the sum of as many terms as the considered points $\mathbf{M}_0 + \sum_i \Delta\mathbf{M}_\mathbf{i} m_{a,i}$. Then our theory assures that the extremal frequencies over arbitrary distributions are indeed achieved by concentrating the mass on a single point.

Therefore we consider the problem of determining the largest interval over all possible choices of such a single point on the floors (and not only on the numerated nodes). In this case, the maximum frequency $f = 9.714$ Hz is obtained by placing the additive mass at the point $[b_x, b_y] = [0.825, 0.825]$ either at the first or at the second floor. Instead, the minimum possible frequency is $f = 2.377$ Hz and it is reached by placing the mass on the second floor on point $[b_x, b_y] = [4, 4]$, which corresponds to node 6 as indicated in Fig. 4 as before.

6.2. Example 2

Let us consider the vertical tensile structure represented in Fig. 6, subjected to the gravity force parallel to the vertical cables. The structure has 3 degrees of freedom (displacements of the planes). The stiffness parameters depend on the cable tension forces, which are caused by the gravity forces and then they depend on the masses as well. As a consequence, a variation of the mass produces not only a variation of the mass matrix but a variation of the stiffness matrix as well. Thus, this structure has the peculiar characteristic that augmenting the mass might also result in increasing some natural frequencies and vice versa.

Assume that a tension T_0 is applied on the cables once all the masses are placed, so that T_0 is the actual tension on the lowest portion of the cable. The mass matrix and the stiffness matrix can thus be easily computed in the following way:

$$\mathbf{M} = \begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix},$$

$$\mathbf{K} = \frac{1}{h}\begin{bmatrix} 4 & -2 & 0 \\ -2 & 4 & -2 \\ 0 & -2 & 4 \end{bmatrix} T_0 + \frac{1}{h}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} m_1 g + \frac{1}{h}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} m_2 g + \frac{1}{h}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} m_3 g.$$

Note that the expression of $\mathbf{K}$ includes three geometric terms associated with the gravity force, hence with the mass and g for the reasons previously explained.

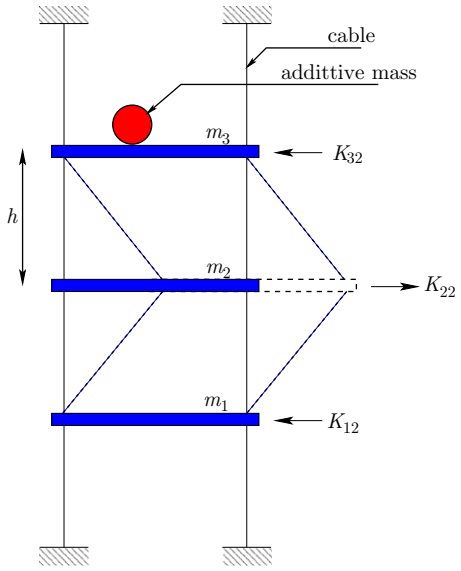

Fig. 6. A simple three DOF's tensile structure.

The total additive mass is $\mu_a = \mu_{a1} + \mu_{a2} + \mu_{a3}$ where μ_{ai} is the portion of the additive mass added to the ith floor. If we assume $m_1 = m_2 = m_3 = 3\,\text{kg}$, $h = 0.5\,\text{m}$, $T_0 = 100\,\text{N}$ and $g = 10\,\text{m/s}^2$, then the mass and the stiffness matrices become

$$\mathbf{M} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu_{a1} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \mu_{a2} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mu_{a3},$$

$$\mathbf{K} = \begin{bmatrix} 860 & -460 & 0 \\ -460 & 980 & -520 \\ 0 & -520 & 1100 \end{bmatrix} + 20 \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mu_{a1} + 20 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} \mu_{a2} + 20 \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mu_{a3}.$$

According to our results, the minimum and the maximum natural frequencies are obtained when $\mu_{a,i} = m_a$ for some i. Figs. 7 and 8 show the behavior of the minimum and maximum natural frequencies as a function of the total additive mass. If this exceeds a certain amount (around 25 kg in the examined cases), positioning the whole mass on the first floor maximizes the third natural frequency and minimizes the first one. Conversely, for additive mass below a certain threshold (around 4.5 kg), both minimum and maximum frequencies are minimized when the additive mass is placed on the second floor. The plot of all natural frequencies is reported in Fig. 9.

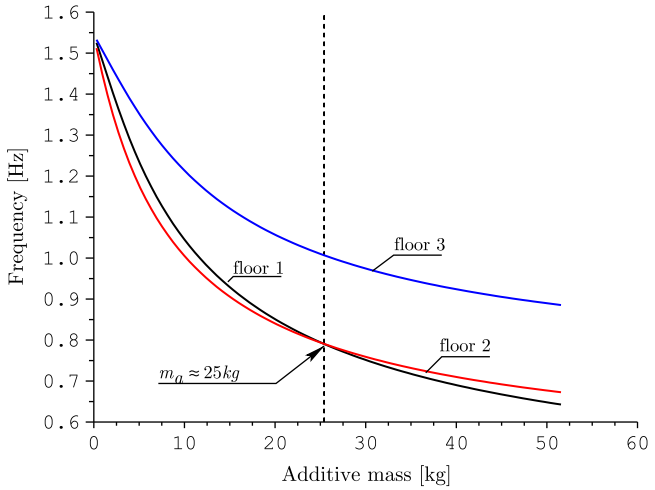

Fig. 7. Minimum natural frequency versus additive mass.

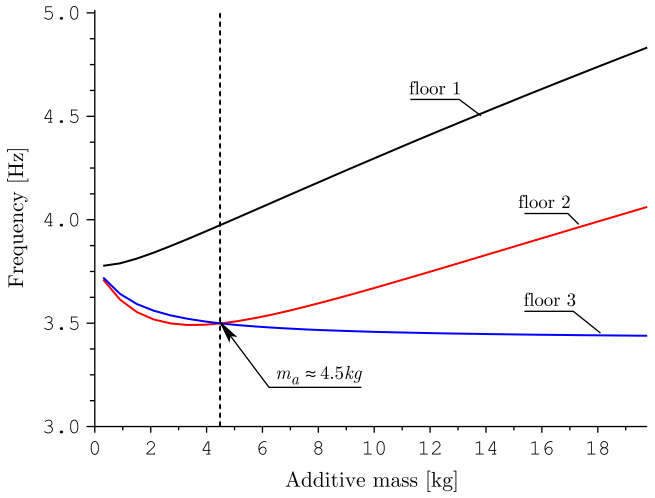

Fig. 8. Maximum natural frequency versus additive mass.

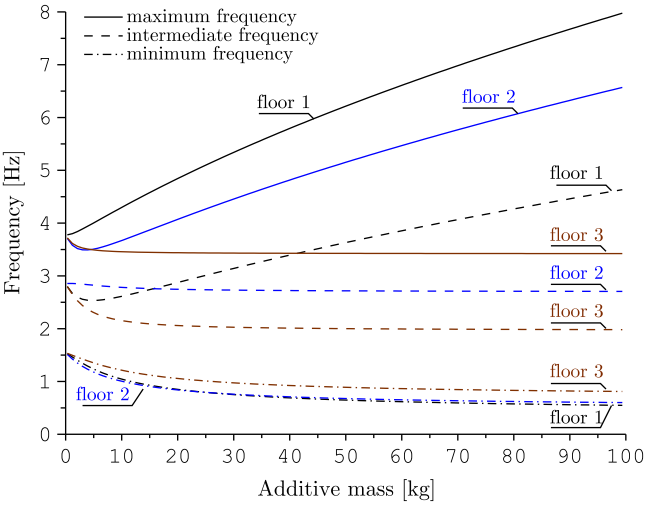

Fig. 9. Spectra of the natural frequencies versus the additive mass.

7. Discussion and conclusions

In this paper we proved that for uncertain vibrating systems whose matrices depend affinely on uncertain parameters which are linearly bounded the weak frequency interval detection problem can be solved by means of a finite number of eigenvalue problems. We have also investigated more general types of uncertainty, for which no general recipes are available up to immersion techniques in which the original system family is replaced by an over-bounding one. The strong interval detection problem can be solved by a finite number of tests under stronger assumptions of the uncertain representation. These results essentially recover those due to previous references [4] as a special case.

In principle, from the computational standpoint, the interval detection problem can be solved by Monte Carlo or optimization methods. However, these have the disadvantages that, typically, the number of samples or the number of iterations can be very large. Besides, they are *approximated methods*. We actually show that for affine systems *at least* the greater and the smaller frequencies can be exactly derived by means of a finite number of tests corresponding to the vertices of the systems. We stress that knowing the upper and the lower frequencies is quite informative in most cases, especially in systems with large uncertainties in which the intermediate frequency intervals are deeply overlapping.

Future work along this lines includes the investigation of other classes of uncertainties, possibly non-affine, or the continuous mass distribution case. We also point out the problem of characterizing relevant quantities associated with the modes such as the eigenvectors and the participating mass defined as

$$\gamma \doteq \mathbf{1}^{\mathrm{T}}\mathbf{M}\mathbf{u}_i/\mathbf{u}_i^{\mathrm{T}}\mathbf{M}\mathbf{u}_i,$$

where $\mathbf{u}_i$ is the normalized eigenvector associated with ith mode and $\mathbf{1}^{\mathrm{T}} = [1\ \ 1\ \ \ldots\ \ 1]$. Clearly the problem is much harder and it is not clear if, under the assumption that $\mathbf{M}$ and $\mathbf{K}$ are affine functions of the uncertain parameters, some kind of interval detection problem can be solved. We were unable to provide results or counterexamples at the moment of writing this paper. Finally, we believe that the polytopic uncertainty description adopted here can be quite useful in solving other problems relevant to elastic structures such as characterizing static and dynamic responses. Further interesting connections with the system theory literature [13,14] are expected in this issue.

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